20/9/23

MATH2050A Tutorial

Announcements:

- HW1 posted on course website (note it is different than the one posted for MATH2050B). Due Mon. 25/9 on Gradescope 11:59pm.

Q1: $K := \{ s + t\sqrt{2} : s, t \in \mathbb{Q} \}$ Show that:

1) if $x_1, x_2 \in K$, then $x_1 + x_2,\ x_1 \cdot x_2 \in K$
2) if $x \neq 0, x \in K$, then $\frac{1}{x} \in K$.

Pf: $x_1 = s_1 + t_1\sqrt{2}$, $x_2 = s_2 + t_2\sqrt{2}$,

$$x_1 + x_2 = s_1 + t_1\sqrt{2} + s_2 + t_2\sqrt{2}$$

commutativity (a1)

$$= s_1 + s_2 + t_1\sqrt{2} + t_2\sqrt{2}$$

(d)

$$= \underbrace{s_1 + s_2}_{\in \mathbb{Q}} + \underbrace{(t_1 + t_2)}_{\in \mathbb{Q}}\sqrt{2} \in K.$$

$$x_1 x_2 = (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) = s_1 s_2 + s_1 t_2\sqrt{2} + s_2 t_1\sqrt{2} + t_1 t_2 \underbrace{\sqrt{2}\sqrt{2}}_{2}$$

$$= s_1 s_2 + 2t_1 t_2 + (s_1 t_2 + s_2 t_1)\sqrt{2} \in K.$$

$x \neq 0 \Rightarrow$ at least $s \neq 0$ or $t \neq 0$, $\qquad \in \mathbb{Q} \qquad \in \mathbb{Q}$

$$\frac{1}{x} = \frac{1}{s+t\sqrt{2}} \cdot \frac{s-t\sqrt{2}}{s-t\sqrt{2}}$$

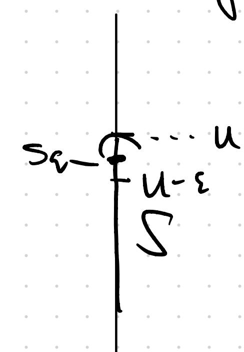
$$= \frac{s-t\sqrt{2}}{s^2-2t^2} = \underbrace{\frac{s}{s^2-2t^2}}_{\in\mathbb{Q}} - \underbrace{\frac{t}{s-2t^2}}_{\in\mathbb{Q}}\sqrt{2} \in K.$$

This shows, that there is a ordered subfield $\mathbb{Q} \subsetneq K \subsetneq \mathbb{R}$

Q2: Show the following 2 statements are equivalent: for S nonempty, u an upper bound of S.

1) if v is any upper bound of S, then $u \leq v$.

2) if $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.



Pf: (1) $\Rightarrow$ (2): Suppose u satisfies (1). let $\varepsilon > 0$ be given.

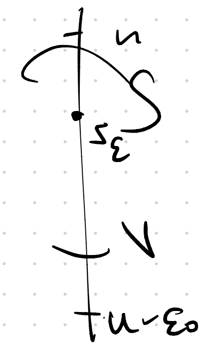
Clearly $u - \varepsilon < u$. So by (1), $u-\varepsilon$ is not an upper bound of S. (Taking contrapositive of (1): $P \Rightarrow Q \iff \neg Q \Rightarrow \neg P$).

Since $u-\varepsilon$ is not an upper bound of S, there exists some $s_\varepsilon \in S$ s.t. $u-\varepsilon < s_\varepsilon$.

(2) $\Rightarrow$ (1): Suppose u satisfies (2). Let v be an upper bound of S. Suppose on the contrary that $v < u$. Taking $\varepsilon_0 = u - v$.

So by (2), there is an $s_\varepsilon \in S$, s.t.

$$S \ni s_\varepsilon > u - \varepsilon_0 = u-(u-v) = v. \quad \text{so } v < s_\varepsilon \in S,$$



which contradicts the fact that v is an upper bound of S.

$\Rightarrow u \leq v$. //

Q3: X, Y non-empty, $h: X \times Y \to (-\infty, \infty)$ have bounded range.

$f: X \to \mathbb{R}$ by $f(x) := \sup\{h(x,y) : y \in Y\}$

$g: Y \to \mathbb{R}$ by $g(y) := \inf\{h(x,y) : x \in X\}$.

Show that $\sup\limits_{y\in Y} g(y) \leq \inf\limits_{x\in X} f(x)$

(above can be written as $\sup\limits_{y} \inf\limits_{x} h(x,y) \leq \inf\limits_{x} \sup\limits_{y} h(x,y)$, this is Problem 2.4.11 of textbook, 2.4.9, 2.4.10 explore examples)

Pf: By definition of f, g, $\forall x \in X, y \in Y$,

$$g(y) \leq h(x,y) \leq f(x).$$

"Take infimum over x": Fix y. Then $g(y) \leq f(x)$. So in particular, by def'n of infimum as greatest lower bound, $\inf\{f(x) : x \in X\}$.

"Take sup over y": $g(y) \leq \inf\limits_{x\in X} f(x) \leq f(x)$.

Since this is true for each y, $\inf\limits_{x\in X} f(x)$ is an upper bound for $\{g(y) : y \in Y\}$, then by def'n of supremum as least upper bound

$$g(y) \leq \sup_{y\in Y} g(y) \leq \inf_{x\in X} f(x).$$